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Percolation of even sites for random sequential adsorption

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Abstract

Consider random sequential adsorption on a red/blue checker-board lattice with arrivals at rate 1 on the red squares and rate λ on the blue squares. We prove that the critical value of λ , above which we get an infinite blue component, is finite and strictly greater than 1.

Key words and phrases: Dependent percolation, random sequential adsorption, critical points.

AMS classifications: 60K35, 82B43

1 Introduction

Random sequential adsorption (RSA) is a term used for a family of probabilistic models for irreversible particle deposition. Particles arrive at random locations onto a surface which is typically taken to be two-dimensional and initially empty, and each particle, once accepted, blocks nearby locations from becoming occupied, thereby causing any subsequent particles arriving

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nearby to be rejected. Both lattice and continuum versions of RSA have been studied extensively in the literature. They are of considerable interest in the physical sciences, for example with regard to the coating of a surface by some adsorbed substance [4, 10].

In the present paper we consider the two-dimensional lattice version of RSA, whereby the surface is represented by \mathbb{Z}^2 endowed with the usual nearest-neighbour graph structure. The arrival times $t_x, x \in \mathbb{Z}^2$, are taken to be independent and exponentially distributed. Initially all sites are vacant, but if a particle arrives at x at time t_x , then site x becomes occupied at that instant unless one of the neighbouring sites was previously occupied. That is, when a particle becomes occupied it causes all of its neighbours to be blocked. Ultimately, every site is either occupied or blocked. Provided there is a uniform bound on the arrival rates, the model is well defined even on the infinite lattice \mathbb{Z}^2 : see e.g. [8] or [9].

The ultimate configuration is called the *jammed state* and is the focus of our attention here. In the jammed state, the occupied lattice sites comprise a maximal stable set (a stable set is a subset of vertices in the graph such that no two vertices in that set are adjacent). The remaining sites are blocked.

Since the \mathbb{Z}^2 lattice is bipartite, the set of occupied sites is naturally partitioned into two phases, the even and odd occupied sites, where a site is denoted even/odd according to its graph distance from the origin. In fact, we can partition the whole of \mathbb{Z}^2 into two phases, one phase consisting of even occupied sites and odd blocked sites (the *even phase*), and the other consisting of odd occupied sites and even blocked sites (the *odd phase*). Since we are in the jammed state, all sites lie in one phase or the other.

We are interested in the *percolation* properties of the even phase. That is, we consider the question of whether the subgraph of \mathbb{Z}^2 induced by the even phase contains an infinite component. Physically, such questions could be of interest with regard to, for example, electrical or thermal conductivity through adsorbed particles on a surface. Percolation properties of particle configurations generated by RSA type processes have been studied in the physical sciences literature; see for example Section VI of [4], [7] and [11], and references therein.

The sites in the even phase form a dependent site percolation process on \mathbb{Z}^2 . A basic result of this paper is that if the arrival rates at all sites are the same, then the even phase will not percolate (and neither will the odd phase). Therefore the odd and even phases decompose into finite connected islands (cf. the diagrams on page 1309 of [4]).

One can tune the model by biasing the arrival rates in favour of the even sites, and this is what we do, with a single parameter λ representing the amount of bias (this version of the model was suggested to us by Martin Zerner). One might expect the even phase to percolate given a sufficiently high level of bias. We shall show that there is a non-trivial phase transition in the parameter λ . In particular, there is a non-zero level of bias at which the even phase still does not percolate. This improves on the aforementioned basic result, and is our main result.

We briefly discuss the degree of surprise in the basic result of non-percolation when all arrival rates are the same. It is known that independent site percolation with parameter $p = 1/2$ on the usual square lattice does not percolate, and the density of the even phase in RSA is $1/2$, suggesting by analogy that our dependent site percolation process would not percolate.

On the other hand, the process of occupied even sites may be viewed as a dependent site percolation process on a square lattice *with the diagonal edges added*. Indeed, if one turns the original lattice through 45 degrees, the even sites form a square grid for which any two neighbours (including diagonal neighbours) must be in the same component of the even phase if they are both occupied, since they have at least one odd neighbour in common in the original lattice.

Site percolation on the square lattice with diagonals is strictly supercritical at $p = 1/2$ (it is dual to site percolation on the usual square lattice which is strictly subcritical), so from this one might expect the even phase of a RSA-type hard-core process (i.e., one which generates a random stable subset of \mathbb{Z}^2) with density sufficiently close to $1/2$ to percolate. However, our results suggest otherwise. While the basic RSA process considered here has a density strictly below $1/2$, it seems likely that RSA can be modified to provide a hard-core process with density of occupied sites arbitrarily close to one-half, without affecting the basic non-percolation result. To see this, consider a variant where, initially, large square blocks of sites arrive sequentially at random locations. When a square block arrives, suppose all sites in the block with the same parity as its lower left corner become occupied, unless one or more of them is already blocked, in which case the entire incoming square block is rejected. At the end of this process there remain some holes, but these can be filled in by having a subsequent arrivals process of smaller square blocks.

On the other hand, we think it is also likely that there exist stationary ergodic hard-core processes on the sites of the square lattice for which the

even phase does percolate almost surely. Indeed, consider a stationary curve along the lines of the one in the proof of Proposition 5 of Holroyd and Liggett [5], and put the odd phase on one side of this and the even phase on the other side.

Van den Berg [1] considers another form of dependent percolation, with biological motivation. That paper is concerned with sharp transitions for percolation on a the random field associated with the contact process, whereas in the present instance we are concerned with inequalities of critical points for a random field generated by random sequential adsorption. It is noteworthy, however, that in both cases the methods of Bollobás and Riordan [2] play a key role.

2 Statement of result

We now describe the model in more detail. Let Λ denote the square lattice with vertex set \mathbb{Z}^2 and edges between each pair x, y with $|x - y| = 1$ (with $|\cdot|$ denoting Euclidean distance). Given a value $\lambda > 0$ of the model parameter, let $(t_x, x \in \mathbb{Z}^2)$ be a family of independent exponential random variables with rate 1 for x odd and rate λ for x even. We shall sometimes refer to t_x as the *arrival time* at x . We write P_λ for probability given a value λ for the model parameter.

Initially all sites are empty. If none of the four neighbours of x are occupied at time t_x we declare x to be occupied from then on. If any of its neighbours becomes occupied then site x becomes blocked at that instant, and remains so from then on. In this way every site will eventually end up being occupied or blocked (see [8] or [9].)

If an even site is occupied we declare it to be black and if it is blocked we declare it to be white. If an odd site is occupied we declare it to be white and if it is blocked we declare it to be black. The black sites form the even phase mentioned earlier. We form a graph of black vertices with edges between any two black vertices that are adjacent in the square lattice. By the ergodic property of any family of independent identically distributed variables indexed by \mathbb{Z}^2 , the probability that there is an infinite black component is either zero or one. Moreover, by a standard coupling argument, this probability is monotonic nondecreasing in λ . Therefore, there is a critical value $\lambda_c \in [0, \infty]$, such that for $\lambda > \lambda_c$ there will almost surely be an infinite black component and for $\lambda < \lambda_c$ there will almost surely not be an infinite

black component. Our main result provides some non-trivial bounds on this critical value.

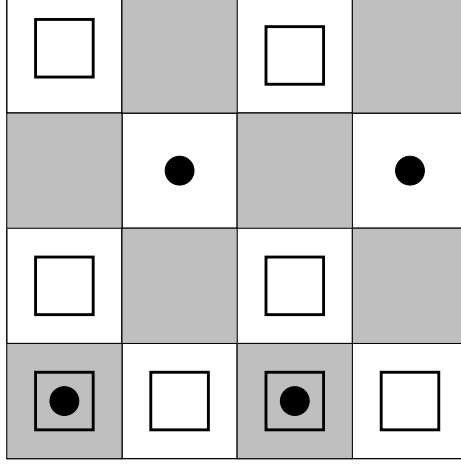


Figure 1: Example: The shaded squares represent even sites. The squares with a circle inside represent occupied sites. The squares with an inscribed square inside represent black sites.

Theorem 2.1 *It is the case that $1 < \lambda_c < 10$.*

Proof of $\lambda_c < 10$. This upper bound is simple to prove, and we deal with it at once. Let $\mathcal{V} \subset \mathbb{Z}^2$ denote the set of all sites that have even coordinates adding up to a multiple of 4, such as $(0, 0)$, $(2, 2)$, $(0, 4)$ and so on. Define an adjacency relation \sim on \mathcal{V} by putting $x \sim x'$ whenever $|x - x'| = 2\sqrt{2}$. The resulting graph (\mathcal{V}, \sim) is a tilted square lattice.

Now deem each site $x \in \mathcal{V}$ to be *open* if $t_x < t_y$ for all $y \in \mathbb{Z}^2$ with $|x - y| = 1$. Then each $x \in \mathcal{V}$ is open with probability $\frac{\lambda}{4+\lambda}$, independently of the other sites in \mathcal{V} . If x is open then it is occupied in the original RSA process, and if two adjacent sites in the lattice \mathcal{V} are open then the even site midway between them will also be occupied in the RSA process (because all of its neighbours are blocked). Therefore if there is an infinite component of open sites in the lattice \mathcal{V} there is also one in the even phase of the original RSA process with parameter λ , so by comparison with independent site percolation on (\mathcal{V}, \sim) we have the following inequality:

$$\frac{\lambda_c}{4 + \lambda_c} \leq p_s$$

where p_s is the critical site probability on the square lattice, which is known to be less than 0.7 (Wierman [12]). Rearranging gives that

$$\lambda_c \leq \frac{4p_s}{1-p_s} < \frac{28}{3} < 10$$

so we have proved the upper bound. \square

In the remaining sections, we shall prove the lower bound $\lambda_c > 1$. Although the result is perhaps to be expected by analogy with known (though non-trivial) results for Bernoulli (i.e., independent) site percolation, we are not aware of any such results in a dependent site percolation setting such as we consider here. By use of the weak RSW-type lemma established by Bollobás and Riordan [2] for percolative systems enjoying weak dependence, we shall rather quickly establish the weak version of the inequality, namely $\lambda_c \geq 1$ (see Remark 4.1). To make this inequality strict we use the technique of enhancement. While this technique is well known, in the present setting its application is quite intricate, requiring several notions of pivotal vertex (see Sections 4 and 5).

3 Duality

Define the dual lattice Λ^* to be the square lattice Λ with the diagonals added so that two sites $x, y \in \mathbb{Z}^2$ are adjacent if $|x - y| = 1$ or $|x - y| = \sqrt{2}$. On any rectangular set of sites we have either a black horizontal crossing in Λ or a white vertical crossing in Λ^* , but not both.

Define $f_\lambda(\rho, s)$ to be the P_λ -probability that there is a horizontal black crossing of the rectangle $[1, 2\lfloor \frac{\rho s}{2} \rfloor] \times [1, 2\lfloor \frac{s}{2} \rfloor]$ (an approximately $\rho s \times s$ lattice rectangle with even side lengths). Define $f_\lambda^*(\rho, s)$ to be the probability that there is a horizontal black crossing of this rectangle when we allow diagonal edges as well.

In subsequent sections, we shall prove the following key result.

Proposition 3.1 *There exists $\mu < 1$ such that*

$$\liminf_{s \rightarrow \infty} f_\mu^*(1, s) > 0. \tag{3.1}$$

In the remainder of the present section, we show how to complete the proof of Theorem 2.1, given Proposition 3.1. The argument uses two further results, which we give now.

We say site $x \in \mathbb{Z}^2$ *affects* site $y \in \mathbb{Z}^2$ if there exists a self-avoiding path in \mathbb{Z}^2 starting at a neighbour of x (in Λ) and ending at y , such that if the odd sites in the path are listed in order as x_1, x_2, \dots, x_m , then $t_{x_1} \leq t_{x_2} \leq \dots \leq t_{x_m}$. If x does not affect y , then any change to t_x (with other arrival times unchanged) will not cause any change to the occupied/blocked status of site y .

Since the event that x affects y is defined only in terms of odd sites, its probability does not depend on the model parameter λ . Later we consider a version of model with different arrival rates at different even sites, but the arrival rates at odd sites are always 1 so the probability that x affects y remains the same. Similarly to arguments in [8], we have the following simple lemma.

Lemma 3.1 *Let $\lambda \in (0, \infty)$, $x \in \mathbb{Z}^2$. Then the P_λ -probability that site x is affected from distance greater than r does not depend on x or λ , and tends to zero as $r \rightarrow \infty$. Likewise, the P_λ -probability that site x affects some site at distance greater than r from x does not depend on x or λ , and tends to zero as $r \rightarrow \infty$.*

Proof. For any self-avoiding path of length r , taking successive odd sites along the path one has at least $\lfloor r/2 \rfloor$ independent identically distributed arrival times, so the probability they occur in increasing order is at most $1/\lfloor r/2 \rfloor!$. Therefore the probability that x is affected from distance greater than r is at most $4(3^r)/\lfloor r/2 \rfloor!$, which tends to zero as $r \rightarrow \infty$. The proof of the second part is similar. \square

We also use the following much deeper lemma, which is a weak version of the RSW lemma for dependent percolation.

Proposition 3.2 *Let $\lambda > 0$ and $\rho > 1$ be fixed. If $\liminf_{s \rightarrow \infty} f_\lambda^*(1, s) > 0$ then $\limsup_{s \rightarrow \infty} f_\lambda^*(\rho, s) > 0$.*

A result along these lines is given by Bollobás and Riordan (Theorem 4.1 of [2]). The result in [2] is for Voronoi percolation but the proof can be transferred to our model, as we now discuss.

Much of the proof of [2, Theorem 4.1] relies only on the Harris-FKG inequality, which holds in the present model as well (see Theorem 5 of Penrose and Sudbury [9]). The proof in [2] proceeds by a series of lemmas and claims, and we describe how to adapt two of these to the present setting. Claim 4.3 of [2] can be adapted to the integer lattice as follows.

Lemma 3.2 *Let $\lambda \in (0, \infty)$ and $\varepsilon \in (0, 1/2)$ be fixed. For $s > 1$, let T_s be the strip $[1, \lfloor s \rfloor] \times \mathbb{Z}$, and let $L(s)$ be the line segment $\{1\} \times [\lfloor -\varepsilon s \rfloor, \lfloor \varepsilon s \rfloor]$.*

If $\limsup_{s \rightarrow \infty} f_\lambda^(1 + \varepsilon, s) = 0$ then the P_λ -probability that there is a black path \mathbf{P} in T_s starting from $L(s)$ and going outside $S'(s) = [1, \lfloor s \rfloor] \times [-\lfloor s/2 + 2\varepsilon s \rfloor, \lfloor s/2 + 2\varepsilon s \rfloor]$ tends to zero as $s \rightarrow \infty$.*

Proof. By symmetry in the line $[1, \lfloor s \rfloor] \times \{0\}$ it suffices to show that the event E that there is a black path \mathbf{P}_1 lying entirely within $S'(s)$ and connecting some site of $L(s)$ to some site at the top of $S'(s)$ has probability tending to zero.

Let E_1 be the event that there is such a path \mathbf{P}_1 lying entirely in the rectangle $R(s)$ given by

$$R(s) = [1, \lfloor s \rfloor] \times [2\lfloor \varepsilon s \rfloor - \lfloor s/2 + 2\varepsilon s \rfloor, \lfloor s/2 + 2\varepsilon s \rfloor],$$

which has its lower boundary given by reflecting the upper boundary of $S'(s)$ in the line $y = \lfloor \varepsilon s \rfloor$, and which has height at least $\lfloor (1 + 2\varepsilon)s - 3 \rfloor$. If E holds but E_1 does not then there is a black crossing the long way of $R(s)$, which has P_λ -probability tending to zero. Therefore it suffices to show that $P_\lambda(E_1) \rightarrow 0$.

Reflecting vertically in the line $y = \lfloor \varepsilon s \rfloor$, let $L'(s) := \{1\} \times [\lfloor \varepsilon s \rfloor, 3\lfloor \varepsilon s \rfloor]$ be the image of $L(s)$. Let E_2 be the event that there is a black path \mathbf{P}_2 from $L'(s)$ to some point with height $2\lfloor \varepsilon s \rfloor - \lfloor s/2 + 2\varepsilon s \rfloor$. Then by symmetry and by the Harris-FKG inequality, the probability that E_1 and E_2 occur is at least $P_\lambda(E_1)^2$. But if $E_1 \cap E_2$ occurs then \mathbf{P}_1 and \mathbf{P}_2 must meet and therefore there is a black path crossing $R(s)$ from top to bottom. This has probability tending to zero, so $P_\lambda(E_1) \rightarrow 0$. \square

In the course of the proof of Theorem 4.1 of [2], an event there denoted E_{dense} is considered, and we need a version of this event here. Given any integer rectangle $R = [a, b] \times [c, d]$, and given also $r \in \mathbb{N}$, let $R[r]$ be the larger rectangle $[a - r, b + r] \times [c - r, d + r]$. Let $E_{dense}(R, r)$ be the event that no site in R is affected by any site outside $R[r]$, and note that $E_{dense}(R, r)$ depends only on the arrival times at sites in $R[r]$. By a similar argument to the proof of Lemma 3.1 we have the following result, which is analogous to Lemma 3.2 of [2].

Lemma 3.3 *Let $\lambda > 0$, $\rho \geq 1$. Given $s > 0$, let $R_s = [1, \lfloor s \rfloor] \times [1, \lfloor \rho s \rfloor]$. Then $P_\lambda[E_{dense}(R_s, 2\lfloor \sqrt{s} \rfloor)] \rightarrow 1$ as $s \rightarrow \infty$.*

Proof of Proposition 3.2. To prove Proposition 3.2, assume for a contradiction that it does not hold and fix a value of λ where it fails. Then $\liminf_{s \rightarrow \infty} f_\lambda^*(1, s) > 0$ and for some $\rho > 1$ we have $\lim_{s \rightarrow \infty} f_\lambda^*(\rho, s) = 0$. Then, as in (4.4) of [2], for any $\varepsilon > 0$ we have $\limsup_{s \rightarrow \infty} f_\lambda^*(1 + \varepsilon, s) = 0$.

We can then follow the proof of [2, Theorem 4.1]. Our Lemmas 3.2 and 3.3 take the place of Claim 4.3 and Lemma 3.2 of [2] respectively; the other claims in the proof of [2, Theorem 4.1] are easily adapted to the current setting, provided we make sure that rectangles with integer sides are chosen since the RSA model is on a discrete lattice rather than a continuum. \square

For $n \in \mathbb{N}$ we define the boxes

$$B(2n+1) := [-n, n] \times [-n, n]; \quad B(2n) := [-n, n-1] \times [-n, n-1]. \quad (3.2)$$

Proof of Theorem 2.1. By Proposition 3.1, there exists $\mu < 1$ such that (3.1) holds. Defining $\delta := (1/3) \limsup_{s \rightarrow \infty} f_\mu^*(4, s)$, we have by (3.1) and Proposition 3.2 that $\delta > 0$.

Thus we can find infinitely many even m such that the probability of a black crossing (including diagonals) the long way of a $4m$ by m integer rectangle is at least 2δ (we take m even so that this probability does not depend on the location of the rectangle).

Given such even m , taking $n = m + 1$ we have for any $3n$ by n rectangle that there is a $4m \times m$ rectangle for which the existence of a long-way crossing of the $4m \times m$ rectangle would imply the existence of a long-way crossing of the $3n \times n$ rectangle (provided $4m \geq 3n$, i.e. $m \geq 3$.)

Therefore we can find infinitely many odd n such that there is a crossing of a $3n$ by n rectangle with probability at least 2δ . Let \mathbf{n} be the set of all such odd n . Then for all $n \in \mathbf{n}$, using the Harris-FKG inequality for this model (see [9]), the probability of there being a circuit of the annulus $B(3n) \setminus B(n)$ is at least $(2\delta)^4$.

By Lemma 3.1, given n we can find an $m > n$ such that

$$P_\mu \left[\bigcup_{y \in \mathbb{Z}^2 \cap B(n), z \in \mathbb{Z}^2 \setminus B(m)} (\{y \text{ affects } z\} \cup \{z \text{ affects } y\}) \right] \leq \delta^4.$$

Thus, we can build up a sequence of positive integers $m_1 < n_1 < m_2 < n_2 < \dots$ such that (i) $n_i \in \mathbf{n}$ for each $i \in \mathbb{N}$, and (ii) $3n_i < m_{i+1}$ for each $i \in \mathbb{N}$, and (iii) the probability that there exists any vertex inside the annulus $A_i := B(3n_i) \setminus B(n_i)$ that is affected from outside the annulus $A'_i := B(m_{i+1}) \setminus B(m_i)$ is at most $2\delta^4$.

Let E_i be the event that (i) there is a closed circuit (of Λ^*) around the origin consisting of black sites in the annulus A_i , and (ii) no site of A_i is affected by any site outside A'_i . Then for all i , $P_\mu[E_i] \geq \delta^4$, and the events E_i are mutually independent, because event E_i is in the σ -algebra generated by the arrival times at sites in A'_i . If any one of the events E_i occurs there cannot be an infinite white component in Λ containing the origin, so by the Borel-Cantelli lemma the probability of an infinite white component occurring is 0. Therefore

$$\lambda_c \geq 1/\mu > 1$$

which completes the proof, subject to proving Proposition 3.1. \square

4 Enhancement

We now define an enhancement that we shall use to interpolate between the RSA models on Λ and on Λ^* . For $z \in \mathbb{Z}^2$ set $z' := z + (1/2, 1/2)$. Let Λ^+ denote the so-called *centred quadratic lattice* (see [6]) whose vertices consist of $\mathbb{Z}^2 \cup \{z' : z \in \mathbb{Z}^2\}$, where for $z, y \in \mathbb{Z}^2$ we put an edge between z, y whenever $|z - y| = 1$ and an edge between z, y' whenever $|z - y'| = \frac{\sqrt{2}}{2}$. We also consider the infinite tessellation of \mathbb{R}^2 with cells centred at each vertex of Λ^+ , where for $z \in \mathbb{Z}^2$ the cell centred at z' is an ℓ_1 ball of radius $1/4$ (i.e., a diamond) and the cell centred at z is an octagon consisting of that part of the unit square centred on z which does not lie in any of the diamonds. See Figure 2. We refer to this tessellation as the *infinite $(4, 8^2)$ tessellation* because the lattice given by the boundaries of the cells is called the $(4, 8^2)$ lattice in [3], page 155. We shall refer to sites z' as *diamond sites* since the diamonds are centred on these sites.

Now consider a certain dependent face percolation model on the infinite $(4, 8^2)$ tessellation, in which each octagon is given the same colour (black or white) as the corresponding site in the random sequential adsorption model with parameter λ , and each of the diamonds is black with probability p (the *enhancement probability*) and white otherwise (independently of everything else). Thus $p = 0$ is equivalent to RSA on Λ and $p = 1$ is equivalent to RSA on Λ^* . We may equivalently view the dependent face percolation model as a site percolation model on Λ^+ .

Let $P_{\lambda,p}$ denote our probability measure for parameter values λ and p . Under $P_{\lambda,p}$, assume we have independent exponential variables $T_x, x \in \mathbb{Z}^2$

(with parameter 1 for odd x and λ for even x) and uniform(0, 1) random variables $T_{x'}, x \in \mathbb{Z}^2$. For $x \in \mathbb{Z}^2$, we set the arrival time t_x to be T_x and we set x' to be black if $T_{x'} < p$ and white otherwise. We call $T_{x'}$ the *enhancement variable* at x' . For later use, let T' be a further exponential random variable with parameter λ , independent of everything else.

In this dependent face percolation model, let H_n denote the event that there is a horizontal black crossing in Λ of a $2n$ by $2n$ square $B(2n)$ (as defined at (3.2)), and set $h(n, \lambda, p) := P_{\lambda, p}(H_n)$. In this model we must have either a horizontal crossing or a vertical white crossing but not both. Also, for $(\lambda, p) = (1, 0.5)$ the probability of both these events must be the same by symmetry so the probability of a horizontal black crossing is 0.5. That is, for any n we have

$$h(n, 1, 0.5) = 0.5. \quad (4.1)$$

Remark 4.1 *By (4.1) and monotonicity, we have $h(n, 1, 1) \geq 0.5$ and therefore (3.1) holds for $\mu = 1$. Hence, by the argument already given in the proof of Theorem 2.1 at the end of Section 3, we have $\lambda_c \geq 1$. The remainder of this paper is concerned with demonstrating that this inequality is strict.*

Now we introduce the idea of a site being *pivotal*. We say that an even site x is pivotal for event H_n if H_n occurs but if we were to change the arrival time t_x from T_x to $T_x + T'$ (leaving other arrival times and enhancement variables unchanged) then H_n would no longer occur. We say that a diamond site x' is pivotal for event H_n if making x' black means H_n occurs but making x' white means it does not. For even $x \in \mathbb{Z}^2$, and for $y \in \mathbb{Z}^2$, define

$$\begin{aligned} \varphi_{\lambda, p}(n, x) &:= P_{\lambda, p}[x \text{ is pivotal for event } H_n]; \\ \varphi_{\lambda, p}(n, y') &= P_{\lambda, p}[y' \text{ is pivotal for event } H_n]. \end{aligned}$$

We have the following proposition (a variant of the Margulis-Russo formula).

Proposition 4.3 *It is the case that*

$$\frac{\partial h(n, \lambda, p)}{\partial \lambda} = (1/\lambda) \sum_{x \in \mathbb{Z}^2: x \text{ even}} \varphi_{\lambda, p}(n, x). \quad (4.2)$$

and

$$\frac{\partial h(n, \lambda, p)}{\partial p} = \sum_{x \in \mathbb{Z}^2} \varphi_{\lambda, p}(n, x'). \quad (4.3)$$

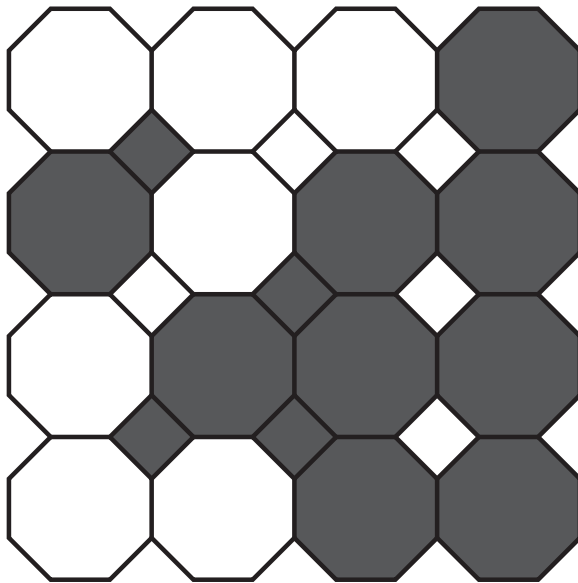
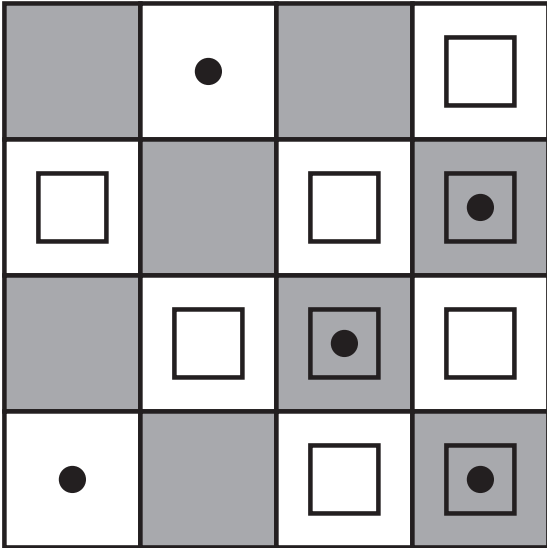


Figure 2: Here is an example of random sequential adsorption and a corresponding percolation process on the faces of the $(4, 8^2)$ lattice

Proof. Fix n, p and λ . Enumerate the even sites of \mathbb{Z}^2 in some manner as x_1, x_2, \dots . Given $k \in \mathbb{N}$ and given $\mu > 0$, let $P_{\lambda,k,\mu}$ denote probability for a model where t_{x_i} (the arrival time at x_i) is exponential with parameter λ for $i = 1, 2, \dots, k-1$ and with parameter μ for $i = k, k+1, k+2, \dots$, and where the enhancement parameter is p (since p is fixed we do not include it in the notation $P_{\lambda,k,\mu}$). For $x \in \mathbb{Z}^2$ let $A(x)$ be the event that site x affects some site in $B(2n)$. By definition and by the proof of Lemma 3.1, $P_{\lambda,k,\mu}[A(x)]$ does not depend on λ, k or μ and decays at least exponentially in the distance from x to $B(2n)$.

Let $\varepsilon > 0$. Assume the exponential variables arising in the definition of the models represented by $P_{\lambda,p}$ and $P_{\lambda,k,\lambda+\varepsilon}$ for all $k \geq 1$, are coupled as follows: for each $i \in \mathbb{N}$ assume we have two independent exponential variables $T_{i,\lambda}$ and $T_{i,\varepsilon}$ with parameter λ and ε respectively. Assume we also have variables T_x (exponential with parameter 1) for odd x and $T_{x'}$ (uniform) for diamond sites x' . For the $P_{\lambda,p}$ model we take $t_{x_i} = T_{i,\lambda}$ for all i while for the $P_{\lambda,k,\lambda+\varepsilon}$ model we take $t_{x_i} = T_{i,\lambda}$ for $i < k$ and $t_{x_i} = \min(T_{i,\lambda}, T_{i,\varepsilon})$ for $i \geq k$. For odd x we take $t_x = T_x$ for all the models, and for each diamond site x' we use the enhancement variable $T_{x'}$ for all the models.

For any set $S \subset \mathbb{Z}^2$, and any $y \in \mathbb{Z}^2 \setminus S$, if x does not affect y for any $x \in S$ then any changes to arrival times ($t_x, x \in S$) with other arrival times unchanged, do not affect the occupied/blocked status of y . Hence,

$$0 \leq P_{\lambda,k,\lambda+\varepsilon}[H_n] - h(n, \lambda, p) \leq P_{\lambda,k,\lambda+\varepsilon} \left[\bigcup_{j=k}^{\infty} A(x_j) \right] \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Hence,

$$\begin{aligned} h(n, \lambda + \varepsilon, p) - h(n, \lambda, p) &= P_{\lambda,1,\lambda+\varepsilon}[H_n] - \lim_{k \rightarrow \infty} P_{\lambda,k,\lambda+\varepsilon}[H_n] \\ &= \sum_{k=1}^{\infty} (P_{\lambda,k,\lambda+\varepsilon}[H_n] - P_{\lambda,k+1,\lambda+\varepsilon}[H_n]). \end{aligned} \quad (4.4)$$

With the exponential variables coupled as described above, $P_{\lambda,k,\lambda+\varepsilon}[H_n] - P_{\lambda,k+1,\lambda+\varepsilon}[H_n]$ is the probability of the event that (i) $T_{k,\varepsilon} \leq T_{k,\lambda}$, and (ii) event H_n occurs if we set $t_{x_k} = T_{k,\varepsilon}$, but not if we set $t_{x_k} = T_{k,\lambda}$. By the memoryless property of exponential random variables, conditional on event (i) the variables $T_{k,\varepsilon}$ and $T_{k,\lambda} - T_{k,\varepsilon}$ are independent exponentials with parameter $\lambda + \varepsilon$ and λ respectively. Therefore

$$P_{\lambda,k,\lambda+\varepsilon}[H_n] - P_{\lambda,k+1,\lambda+\varepsilon}[H_n] = (\varepsilon/(\lambda + \varepsilon))P[F_k(\lambda, \lambda + \varepsilon)] \quad (4.5)$$

where for $\mu > 0$, $F_k(\lambda, \mu)$ denotes the event that H_n occurs if we use the first arrival at x_k but not if we use the second arrival at x_k , with the arrival times t_{x_j} are taken to be exponential with rate λ for $j < k$, and rate μ for $j > k$, and the first arrival at x_k is taken to be exponential with rate μ but the time from the first arrival to the second arrival at x_k is exponential with rate λ .

Now couple events $F_k(\lambda, \lambda + \varepsilon)$ and $F_k(\lambda, \lambda)$ in a similar manner to that already described; that is, for each $i \geq k$ assume the exponential variable with parameter $(\lambda + \varepsilon)$ appearing in the definition of $F_k(\lambda, \lambda + \varepsilon)$ is obtained as $\min(T_{i,\lambda}, T_{i,\varepsilon})$ and in the definition of $F_k(\lambda, \lambda)$ let it be replaced by $T_{i,\lambda}$. Then for any integer $K > n$ we have the event inclusion

$$F_k(\lambda, \lambda + \varepsilon) \triangle F_k(\lambda, \lambda) \subset \left(\bigcup_{x \in \mathbb{Z}^2 \setminus B(2K)} A(x) \right) \cup \left(\bigcup_{\{j \geq k: x_j \in B(2K)\}} \{T_{j,\varepsilon} < T_{j,\lambda}\} \right).$$

For any fixed K the probability of event $\bigcup_{\{j \geq k: x_j \in B(2K)\}} \{T_{j,\varepsilon} < T_{j,\lambda}\}$ vanishes as $\varepsilon \downarrow 0$, while the probability of $\bigcup_{x \in \mathbb{Z}^2 \setminus B(2K)} A(x)$ is independent of ε and vanishes as $K \rightarrow \infty$. Hence by (4.5),

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} (P_{\lambda,k,\lambda+\varepsilon}[H_n] - P_{\lambda,k+1,\lambda+\varepsilon}[H_n]) = \lambda^{-1} P[F_k(\lambda, \lambda)] = \lambda^{-1} \varphi_{\lambda,p}(n, x_k).$$

Moreover, $P[F_k(\lambda, \lambda + \varepsilon)]$ is bounded by the probability that x_k affects some site in $B(2n)$, which is independent of ε and bounded by a summable function of k . Therefore by (4.4), (4.5) and dominated convergence we have

$$\frac{\partial^+ h}{\partial \lambda} = \lim_{\varepsilon \downarrow 0} \frac{h(n, \lambda + \varepsilon, p) - h(n, \lambda, p)}{\varepsilon} = \lambda^{-1} \sum_{k=1}^{\infty} \varphi_{\lambda,p}(n, x_k). \quad (4.6)$$

By a similar argument (we omit details), one can obtain the same expression for the left derivative $\frac{\partial^- h}{\partial \lambda}$. Therefore (4.2) is proven.

The proof for the second part (4.3) is similar. \square

5 Proof of Theorem 2.1

Let $(\lambda, p, n, y) \in (0, \infty) \times (0, 1) \times \mathbb{N} \times \mathbb{Z}^2$, with y even. In Lemma 5.2 and Proposition 5.1 below, we shall estimate $\varphi_{\lambda,p}(n, y)$ in terms of $\varphi_{\lambda,p}(n, z')$ for suitably chosen $z = z(y, n) \in \mathbb{Z}^2 \cap B(2n)$. Then using Proposition 4.3, starting from (4.1) we shall argue that we can compensate for reducing λ slightly below 1 by increasing p above $1/2$, and deduce Proposition 3.1.

We shall need some preliminary lemmas. The first of these helps us to deal with the dependency between the state of different sites in the RSA model. For $r, s \in \mathbb{N}$ with $r \leq s$, Let $C_r := y + B(2r + 1)$ be the square of side $2r + 1$ centred at y , and define the annulus $A_{r,s} := C_s \setminus C_r$.

We shall consider a coupling of two enhanced RSA processes, both with parameter (λ, p) . Let S_x be the arrival times and enhancement variables in one process (so if $x \in \mathbb{Z}^2$ then S_x is exponentially distributed and $S_{x'}$ is a uniformly distributed enhancement variable). Let T_x be the arrival times and enhancement variables in another independent process. Given $r, s \in \mathbb{N}$ with $s \geq r$, we use these to create a third process of arrival times and enhancement variables $U_x^{(r,s)}$, as follows. Put

$$U_x^{(r,s)} := \begin{cases} S_x, & x \notin C_s \text{ or } x \in A_{r,s} \setminus \mathbb{Z}^2 \\ \mathcal{B}_x S_x + (1 - \mathcal{B}_x) T_x, & x \in A_{r,s} \cap \mathbb{Z}^2 \\ T_x, & x \in C_r \end{cases} \quad (5.1)$$

where the \mathcal{B}_x are independent Bernoulli variables with parameter 0.5. In other words, to get from the S_x process to the U_x process, we re-sample all the variables indexed inside C_r , none of the variables indexed outside C_s or at diamond sites in $A_{r,s}$, and a uniformly randomly selected collection of the variables indexed in $A_{r,s} \cap \mathbb{Z}^2$.

The next lemma establishes a sort of conditional independence between the occupancy status, in the $U_x^{(r,s)}$ process, of sites inside C_r and of sites outside C_s , conditional on the occurrence of a certain event associated with sites in the annulus $A_{r-2,s}$.

For $x \in \mathbb{Z}^2$, define $I_S(x)$ to be 1 if site x is occupied and 0 if it is blocked in the (S_x) -process. Similarly, define $I_U^{r,s}(x)$ to be 1 if site x is occupied and 0 if it is blocked in the $(U_x^{(r,s)})$ -process. Define the following sets of sites:

$$M^{(r,s)} := \{x \in A_{r,s} \cap \mathbb{Z}^2 : I_S(x) = 1\}; \quad N^{(r,s)} := \mathbb{Z}^2 \cap A_{r,s} \setminus M^{(r,s)}; \quad (5.2)$$

$$\begin{aligned} M_1^{(r,s)} &:= \{x \in M^{(r,s)} : S_x \leq 1\}; & M_2^{(r,s)} &:= M^{(r,s)} \setminus M_1^{(r,s)}; \\ N_1^{(r,s)} &:= \{x \in N^{(r,s)} : S_x \leq 1\}; & N_2^{(r,s)} &:= N^{(r,s)} \setminus N_1^{(r,s)}. \end{aligned}$$

Define the event

$$\begin{aligned} E_1^{(r,s)} &:= \cap_{x \in M_1^{(r,s)} \cup N_2^{(r,s)}} \{\mathcal{B}_x = 1\} \cap \cap_{x \in M_2^{(r,s)} \cup N_1^{(r,s)}} \{\mathcal{B}_x = 0\} \\ &\quad \cap \cap_{x \in M_2^{(r,s)}} \{T_x \leq 1\} \cap \cap_{x \in N_1^{(r,s)}} \{T_x > 1\} \\ &\quad \cap \cap_{x \in M^{(r-2,r)}} \{T_x \leq 1\} \cap \cap_{x \in N^{(r-2,r)}} \{T_x > 1\}. \end{aligned} \quad (5.3)$$

Lemma 5.1 *Suppose $r, s \in \mathbb{N}$ with $r \geq 3$ and $s \geq r + 3$. If $E_1^{(r,s)}$ occurs then $I_U^{r,s}(x) = I_S(x)$ for all $x \in \mathbb{Z}^2 \setminus C_r$.*

Proof. Assume event $E_1^{(r,s)}$ occurs. Let us start off with all the arrival times given by the S_x process. Then change the arrival times from S_x to T_x at sites $x \in M_2^{(r,s)}$ one by one. Each time we are making the arrival time at an occupied site earlier, so we cannot change the state (occupied or blocked) of any site. Next, change the arrival times in $N_1^{(r,s)}$ one by one. Each time we are making the arrival time at a blocked site later so we cannot change the state of any site. We then have our U_x process on $\mathbb{Z}^2 \setminus C_r$.

Now we change the arrival times for the sites inside C_r . Every site $x \in M^{(r-1,s-1)}$ has $U_x^{(r,s)} \leq 1$ and has all its Λ -neighbours z with $U_z^{(r,s)} > 1$, so is occupied in the $(U_x^{(r,s)})$ -process. Also, every site $z \in N^{(r,s-2)}$ has $U_z^{(r,s)} > 1$ and has at least one occupied neighbour x with $U_x^{(r,s)} \leq 1$, so is vacant.

Thus when we change the arrival times for the sites inside C_r , the states of sites in $A_{r,s-2}$ do not change and therefore the states of sites in $\mathbb{Z}^2 \setminus C_{s-2}$ also do not change. Hence, whatever arrival times we have on C_{r-2} , the states of the sites $x \in \mathbb{Z}^2 \setminus C_r$ do not change, so $I_U^{r,s}(x) = I_S(x)$ for all such x . \square

Recall that y is said to be pivotal for event H_n if this event occurs when we use arrival time $t_y = T_y$ but not when we use $t_y = T_y + T'$. In the next lemma we bound the probability $\varphi_{\lambda,p}(n, y)$ that y is pivotal for H_n , in terms of a series of events that are more manageable in terms of modifying them to make z' pivotal. These events are defined as follows.

For $r \in \mathbb{N}$, let $E(n, y, r)$ denote the event that if we use $t_y = T_y + T'$ then (i) event H_n occurs if we change the colour of all sites in C_r to ‘black’ (leaving unchanged the colour of sites outside C_r) and (ii) event H_n does not occur if we change the colour of all sites in C_r to ‘white’.

Lemma 5.2 *There exists a constant $K_1 \in (0, \infty)$ such that for all $n \in \mathbb{N}$, $(\lambda, p) \in [0.5, 1.5] \times [0.2, 0.8]$ and all even $y \in \mathbb{Z}^2$, we have*

$$\varphi_{\lambda,p}(n, y) \leq P_{\lambda,p}[E(n, y, 1)] + \sum_{r=1}^{\infty} \frac{K_1^r P_{\lambda,p}[E(n, y, r+1)]}{[r/2]!}. \quad (5.4)$$

Proof. Fix even $y \in \mathbb{Z}^2$. For $r \in \mathbb{N}$, let $\tilde{E}(n, y, r)$ be the event that (i) y is pivotal for event H_n , and (ii) event H_n occurs when we use the arrival time $t_y = T_y + T'_y$ but then change the colour of all sites in C_r to ‘black’.

Clearly $\tilde{E}(n, y, r) \subset \tilde{E}(n, y, r+1)$ for all r , and $\cup_{r=1}^{\infty} E(n, y, r)$ is the event that y is pivotal for H_n . Hence we have

$$\varphi_{\lambda,p}(n, y) = P_{\lambda,p}[\tilde{E}(n, y, 1)] + \sum_{r=1}^{\infty} P_{\lambda,p}[\tilde{E}(n, y, r+1) \setminus \tilde{E}(n, y, r)].$$

Therefore, it suffices to prove that there is a constant K_1 such that

$$P_{\lambda,p}[\tilde{E}(n, y, 1)] \leq P_{\lambda,p}[E(n, y, 1)]; \quad (5.5)$$

$$P_{\lambda,p}[\tilde{E}(n, y, r+1) \setminus \tilde{E}(n, y, r)] \leq \frac{K_1^r P_{\lambda,p}[E(n, y, r+1)]}{[r/2]!}, \quad r \geq 1. \quad (5.6)$$

Let $r \in \mathbb{N}$. First we claim that $\tilde{E}(n, y, r) \subset E(n, y, r)$. Indeed, if $\tilde{E}(n, y, r)$ occurs and we use $t_y = T_y + T'$, then H_n does not occur since y is pivotal, so H_n still does not occur after changing all sites in C_r to white; hence $E(n, y, r)$ occurs. This justifies the claim so in particular (5.5) holds.

Now let $F(r)$ be the event that y affects some site outside C_r . We claim that if $\tilde{E}(n, y, r+1)$ occurs but $F(r)$ does not, then $\tilde{E}(n, y, r)$ occurs. This is because in this case, if we put $t_y = T_y$ then event H_n occurs (because event $\tilde{E}(n, y, r+1)$ implies y is pivotal), and if we then change t_y to $T_y + T'$ then black sites outside C_r will remain black (because $F(r)$ does not occur so changes to t_y do not change the colour of sites outside C_r) and thus event $\tilde{E}(n, y, r)$ occurs.

By using both of the preceding claims, we obtain

$$P_{\lambda,p}[\tilde{E}(n, y, r+1) \setminus \tilde{E}(n, y, r)] \leq P_{\lambda,p}[E(n, y, r+1) \cap F(r)]. \quad (5.7)$$

Also, as in the proof of Lemma 3.1 we have

$$P_{\lambda,p}[F(r)] \leq \frac{4(3^r)}{[r/2]!} \quad (5.8)$$

and $F(r)$ depends only on the arrival times inside C_{r+1} . However, it is not independent of $E(n, y, r+1)$.

Now fix r and consider the independent families of arrival times (S_x) and (T_x) , and a coupled arrival time process $U_x^{(r+1, r+4)}$ as defined by (5.1). Let E^S , respectively E^U , be the event that $E(n, y, r+1)$ occurs based on the S_x process, respectively the $U_x^{(r+1, r+4)}$ process. Let F^S , respectively F^U be the event that $F(r)$ occurs based on the S_x process, respectively the $U_x^{(r+1, r+4)}$

process. Then, defining event $A := E_1^{(r+1, r+4)}$ as given by (5.3), we have from Lemma 5.1 the event identity $E^S \cap A = E^U \cap A$. Hence,

$$\begin{aligned} P_{\lambda, p}[E^S \cap F^S] P_{\lambda, p}[A | E^S \cap F^S] &= P_{\lambda, p}[E^S \cap F^S \cap A] \\ &= P_{\lambda, p}[E^U \cap F^S \cap A] \\ &\leq P_{\lambda, p}[E^U \cap F^S] = P_{\lambda, p}[E^U] P_{\lambda, p}[F^S], \end{aligned}$$

where the last identity follows since by (5.1) the $(U_x^{(r+1, r+4)})$ -process is independent of $(S_x, x \in B_{r+1})$. Also, there is a constant K_2 such that

$$P_{\lambda, p}[A | E^S \cap F^S] \geq K_2^{-r}, \quad r \geq 1, (\lambda, p) \in [0.5, 0.5] \times [0.2, 0.8].$$

Combining these inequalities and using the fact that $P_{\lambda, p}[E^U] = P_{\lambda, p}[E^S]$ yields

$$P_{\lambda, p}[E^S \cap F^S] \leq K_2^r P_{\lambda, p}[E^S] P_{\lambda, p}[F^S],$$

and combined with (5.7) and (5.8) this gives us the desired result (5.6). \square

We now define the $z(n, y)$ mentioned at the start of this section.

Definition 5.1 *Given $n \in \mathbb{N}$ with $n \geq 4$, and given even $y \in \mathbb{Z}^2$, let $z(n, y)$ be the nearest even site in $B(2(n-3))$ to y (using Euclidean distance). If there is a choice of two, take $z(n, y)$ to be the first one according to the lexicographic ordering. Let $z'(n, y) := z(n, y) + (1/2, 1/2)$.*

Thus if $y \in B(2(n-3))$ then $z(n, y) = y$. Otherwise, $z(n, y)$ is a site on the boundary of $B(2(n-3))$. In all cases $z'(n, y) \in B(2n)$. The following proposition is a key step in the proof of Theorem 2.1.

Proposition 5.1 *There exists a constant K_3 such that for any $(\lambda, p, n, y, r) \in [0.5, 1.5] \times [0.2, 0.8] \times \mathbb{N} \times \mathbb{Z}^2 \times \mathbb{N}$ with y even, $n \geq 60$, we have that*

$$P_{\lambda, p}[E(n, y, r)] \leq K_3^r \varphi_{\lambda, p}(n, z'(n, y)) \mathbf{1}_{B(2(n+r))}(y) \quad (5.9)$$

where $\mathbf{1}_A(y) = 1$ if $y \in A$ and $\mathbf{1}_A(y) = 0$ otherwise, for any $A \subset \mathbb{Z}^2$.

We shall prove Proposition 5.1 using the following lemma. Given $r \in \mathbb{N}$, we consider for a while the process $U_x := U_x^{(r+32, r+35)}$ as defined by (5.1). Let D_r be the diamond consisting of sites that are at ℓ_1 distance at most r from y . Let G_r be the octagonal region $C_{r+30} \cap D_{2r+49}$, a sort of truncated

square. Note that each of the inner diagonal boundaries of G_r consists of odd sites and is of length 11. The exact length is not important; we just need a reasonably large separation between each corner of the octagon G_r . Let G_r^- be the slightly smaller octagonal region $C_{r+26} \cap D_{2r+49}$.

Lemma 5.3 *There exists a constant $\beta \in (0, \infty)$ with the following property. Given integers $n, r \in \mathbb{N}$ with $n \geq 60$, and given even $y \in \mathbb{Z}^2$ with $y \in B(2(n+r))$, if the event $E(n, y, r)$ occurs in the S_x process, then there exists a stable set $Q_1 \subset G_r \cap \mathbb{Z}^2$ having no element adjacent to the occupied \mathbb{Z}^2 sites of the S_x process outside G_r , and disjoint sets Q_2, Q_3 of diamond sites inside G_r , such that (i) each of Q_1, Q_2, Q_3 has at most βr elements, and (ii) if, in the U_x process, all the sites in Q_1 are occupied, all diamonds in Q_2 are black, all the diamonds in Q_3 are white, and all sites outside G_r are in the same state as for the S_x process, then $z'(n, y)$ is pivotal for the U_x process.*

Proof. For now we assume C_{r+30} (and hence G_r) is contained in $B(2n)$ (so in particular $z(n, y) = y$). Suppose $E(n, y, r)$ occurs; then there must be disjoint black paths in the S_x process up to $\mathbb{Z}^2 \cap C_{r+1}$ from the left and right sides of $B(2n)$. The strategy of the proof is to extend these paths in towards y , possibly modifying them inside G_r while keeping them disjoint in order to make y' pivotal.

Let V be the set of black vertices (for the S_x process) in $B(2n) \setminus G_r$ that are connected to the left hand side of $B(2n)$ by a black path of the S_x process, without using any sites in G_r . Let v be the first even site inside G_r (according to the lexicographic ordering) that is occupied (for the S_x process) and connects to V either directly or via blocked odd sites adjacent to itself and V (and possibly also a black diamond site). Let W be the set of black sites (for the S_x process) in $B(2n) \setminus G_r$ that are connected to the right hand side of $B(2n)$ by a black path of the S_x process that avoids G_r . Let w be the first even site that is occupied inside G_r and connects to W . We now try and build paths from v and w in towards y to make y' pivotal. We consider various cases of where v and w are:

Case 1: Suppose v and w are well away from each other. In this case we can always make y' pivotal. For example, if v and w are as in Figure 3, we can form disjoint paths $\mathbf{P}_1, \mathbf{P}_2$ of even sites in towards y . In this and subsequent diagrams, the chequerboard squares are centred at sites of \mathbb{Z}^2 and are shaded for even sites. Let I be the set of even sites $\{v, w\} \cup \mathbf{P}_1 \cup \mathbf{P}_2$. Let

J be the set of odd sites in $G_r \setminus G_r^-$ that are not adjacent to any site in I or to any of the occupied sites outside G_r . Let J' be the set of odd sites in G_r^- that are three steps (in Λ) away from I . Set $Q_1 := I \cup J \cup J'$. If the sites in Q_1 are occupied for the T_x process, then y' is pivotal. The number of sites in Q_1 is bounded by a constant times r .

In general, if we have v on a horizontal or vertical edge of G_r , then (see Figure 4) we can make the even site at position A in relation to v occupied to start \mathbf{P}_1 , switch the enhancement on at C' . Due to the odd sites at B being occupied this cannot complete a crossing of $B(2n)$.

If v lies beside a diagonal edge of G_r , then (see Figure 5) we can make the even site at position A in relation to v occupied to start \mathbf{P}_1 , switch the enhancement on at C' and due to the odd sites at B being occupied this cannot complete a crossing of $B(2n)$.

Case 2: Suppose v and w are near each other but on a straight edge. If their columns are at distance 4 or more from each other and neither is in position I (see Figure 6) then there is no problem. Their columns cannot be at distance 2 from each other since then v and w would be connected to each other via black sites. If they are at distance 3 then there is no problem as long as neither v nor w is at position I (see Figure 6.) We have the enhancement switched off at D and then extend the paths in towards y .

Case 3: Now suppose v and w are near each other on a diagonal edge. If their diagonals are at distance 3 there is no problem. Their diagonals cannot be at distance 1 as then they would not be disjoint. If their diagonals are at distance 2 and neither is at J (see Figure 7), there is no problem. We have the enhancement switched off at D and switched on at F .

Case 4: Suppose v and w lie near to each other but on a corner. We need to consider possible cases when v is at I or J (see Figure 8).

(a) v is at J . If w is 3 or more diagonals away then there is no problem. If w is 4 or more columns away then there is no problem. This just leaves three possibilities.

(i) w is at M (of Figure 8). Then refer to Figure 9. We can have an occupied even site at E , connected to v via a diamond site. There is no problem unless there is an occupied even site at A that is in W . Then we need to have an occupied odd site at D and have the enhancement at F' switched off. We can make D occupied because we know B is unoccupied

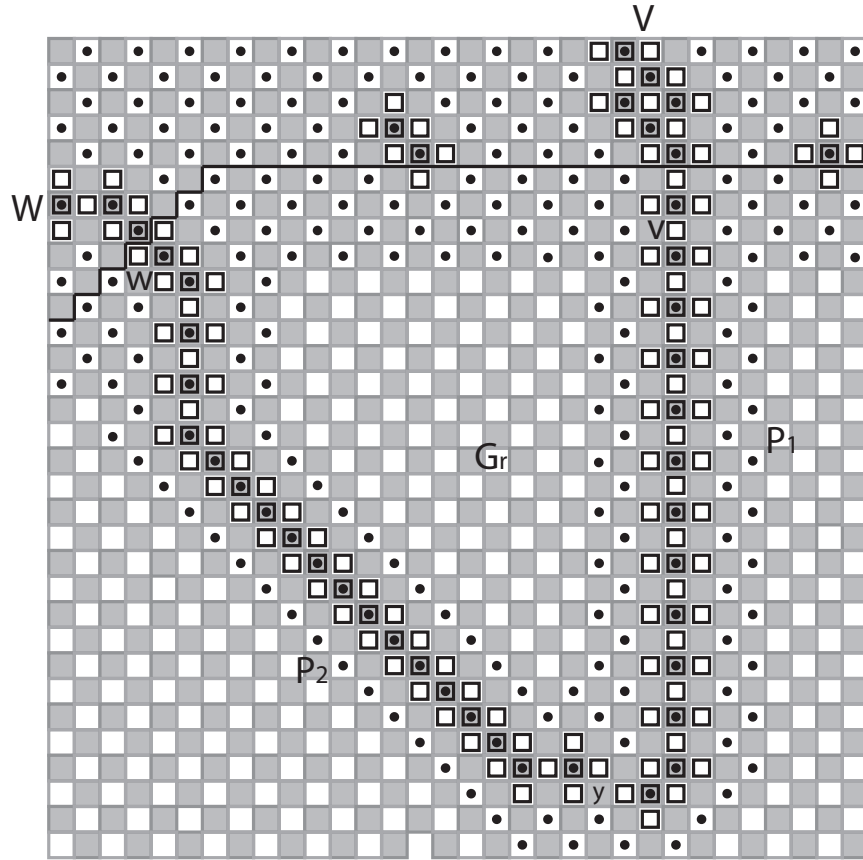


Figure 3: Construction of paths P_1 , P_2 making y' pivotal.

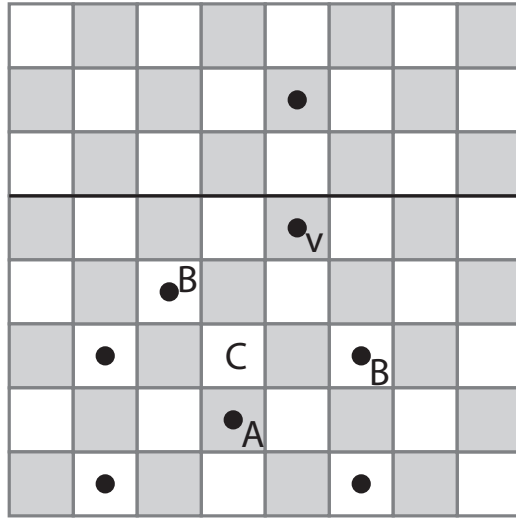


Figure 4: Starting path \mathbf{P}_1 when v is on a horizontal edge on the inner perimeter of G_r .

since otherwise it would connect to both v and W .

(ii) w is at L of Figure 8. In this case, refer to Figure 10. We can have w connected to A and v connected to B , both via enhanced diamond sites, with the enhancement at C' switched off.

(iii) w is at K of Figure 8. Then refer to Figure 11. We aim to have an occupied site at E connected to v . This is fine as long as there is no site of W at B or C . If there is a site of W at C but not B then we need to have an occupied odd site at A and switch off the enhancement, which we can do as we know there is no occupied site at D as it would be joined to v and W . If there is a site of W at B then it is not actually possible to have $E(n, y, r + 1)$ occur since there is no way to get a path from V into D_r without joining up with W . This is because v is blocked from having a path further into G_r without connecting to W , and there cannot be any other point in G_r connected to V elsewhere, because the paths in W from locations in G_r on both sides of v cut v off from being path-connected to any other part of the boundary of G_r .

(b) v is at I of Figure 8. If w is 3 or more diagonals away then there is no problem. If w is 4 or more columns away then there is no problem. This just leaves two possibilities.

(i) If w is at O of Figure 8, then (see Figure 12) this is akin to case (a) (iii) but just translated.

(ii) If w is at N of Figure 8, then (see Figure 13) we aim to connect v to an occupied even site at A . We can do this unless there is an occupied site at B which is in W . If this happens then we aim for an occupied even site at E instead. This works so long as there is no occupied site at C in W . So there is no problem unless there are occupied sites at both B and C in W . If this happens then it is not actually possible to have $E(n, y, r + 1)$ occur since there is no way to get a path from V into D_r without joining up with W .

Now consider the cases where C_{r+30} (and hence G_r) is not contained in $B(2n)$. First we look at the case where C_{r+30} intersects just the top edge of $B(2n)$. We define an octagonal region F_r as follows. Start with the rectangular region $C_{r+30} \cap B(2n)$, which has height at least 30 because we assume $y \in B(2(n + r))$. Then remove triangular regions at the corners to make an octagon. The triangular regions are of height 10 or 11, chosen in such a way that the inner boundary consists of odd sites. We then argue as before using F_r instead of G_r , only now we build our paths in to $z(n, y)$ (which might not be the same as y now) rather than to y . We have the sets

V and W as before and the sites v and w . If v and w are both well away from the edge of $B(2n)$ then we just have one of the cases we have already looked at. So we just consider the case where v say is near the edge of $B(2n)$. However as it is on a diagonal of F_r we can treat it as before and the path we create will stay inside $B(2n)$.

Now consider the case where C_{r+30} intersects the right hand edge of $B(2n)$ but not the left hand edge, and define an octagonal region F_r inside $B(2n)$ analogously to the previous case. However, if $C_{r+30} \cap B(2n)$ shares a corner with $B(2n)$, then do not smooth that particular corner (so in this case F_r is a heptagon). Then $z(n, y)$ will lie in the region F_r . In this case we just look at the set V and site v inside F_r that is connected by a path to the left of $B(2n)$. Inside F_r we can then form a path from v towards $z(n, y)$ and a disjoint path from the right hand edge of $B(2n)$ towards $z(n, y)$ and ensure that $z'(n, y)$ is pivotal.

The case where C_{r+30} intersects the left hand edge of $B(2n)$ but not the right hand edge is treated analogously. In the case where C_{r+30} intersects both the left and the right edges of $B(2n)$, we have $r + 30 \geq n$ so $r \geq n/2$, and we can make a path of even sites in from each boundary of $B(2n)$ to $z(n, y)$, together with a path of odd sites around the edge of each of these paths and around the boundary of B_n . \square

Proof of Proposition 5.1. Assume $y \in B(2(n+r))$; otherwise $E(n, y, r)$ cannot occur. Assume $(\lambda, p) \in [0.5, 1.5] \times [0.2, 0.8]$. Suppose $E(n, y, r)$ occurs for the S_x process. Let the sets Q_1, Q_2, Q_3 be as in Lemma 5.3. Suppose also that $E_1^{(r+32, r+35)}$ occurs, and we have $T_x \leq 1$ on all occupied sites (for the S_x -process) in $C_{r+32} \setminus G_r$ and $T_x > 1$ on all blocked sites (for the S_x -process) in $C_{r+32} \setminus G_r$ (this is consistent with occurrence of event $E_1^{(r+32, r+35)}$.) Suppose also that $T_x \leq 1$ for all the sites in Q_1 and $T_x > 1$ on all the sites in \mathbb{Z}^2 lying adjacent to Q_1 , and $T_{x'} < p$ for $x' \in Q_2$ and $T_{x'} > p$ for $x' \in Q_3$. Then using Lemma 5.1 we have that $z'(n, y)$ is pivotal for the U_x process. This all occurs with probability at least K_3^{-r} (given $E(n, y, r)$), for some finite positive constant K_3 independent of r . Therefore $\varphi_{\lambda, p}(n, z'(n, y)) \geq K_3^{-r} P_{\lambda, p}[E(n, y, r)]$, which completes the proof. \square

Proof of Proposition 3.1. For $n \geq 40$ and even $z \in B(2(n-3))$, set $L_{n, z} := \{y \in \mathbb{Z}^2 : y \text{ even}, z(n, y) = z\}$. For all such n, z , and for $r \in \mathbb{N}$, the set $L_{n, z} \cap B(2(n+r))$ has at most $(r+5)^2$ elements. By Lemma 5.2

and Proposition 5.1, there is a constant K_5 such that for any $(n, \lambda, p) \in \mathbb{N} \times [0.5, 1.5] \times [0.2, 0.8]$,

$$\begin{aligned} \sum_{y \in L_{n,z}} \varphi_{\lambda,p}(n, y) &\leq \sum_{y \in L_z} \left(P_{\lambda,p}[E(n, y, 1)] + \sum_{r=1}^{\infty} \frac{K_1^r P_{\lambda,p}[E(n, y, r+1)]}{[r/2]!} \right) \\ &\leq 36K_3 \varphi_{\lambda,p}(n, z') + \sum_{r=1}^{\infty} \frac{K_1^r K_3^{r+1} (r+6)^2 \varphi_{\lambda,p}(n, z')}{[r/2]!} \leq K_5 \varphi_{\lambda,p}(n, z'). \end{aligned}$$

Summing over even $z \in B(2(n-3))$, we obtain that

$$\sum_{y \in \mathbb{Z}^2: y \text{ even}} \varphi_{\lambda,p}(n, y) \leq K_5 \sum_{z \in B(2(n-3)) \cap \mathbb{Z}^2: z \text{ even}} \varphi_{\lambda,p}(n, z').$$

Hence by Proposition 4.3,

$$\frac{\partial h(n, \lambda, p)}{\partial \lambda} \leq 2K_5 \frac{\partial h(n, \lambda, p)}{\partial p}, \quad (n, \lambda, p) \in \mathbb{N} \times [0.5, 1.5] \times [0.2, 0.8], n \geq 60.$$

We also know from (4.1) that $h(n, 1, 0.5) = 0.5$, so by the Mean Value Theorem, setting $\varepsilon = 0.3/(2K_5)$ we have for all n that

$$h(n, 1 - \varepsilon, 1) \geq h(n, 1 - \varepsilon, 0.8) \geq h(n, 1, 0.5) = 0.5.$$

Therefore taking $\mu = 1 - \varepsilon$ we have (3.1). \square

With Proposition 3.1 proven, our proof of Theorem 2.1 is now complete by the arguments already given in Sections 1 and 3.

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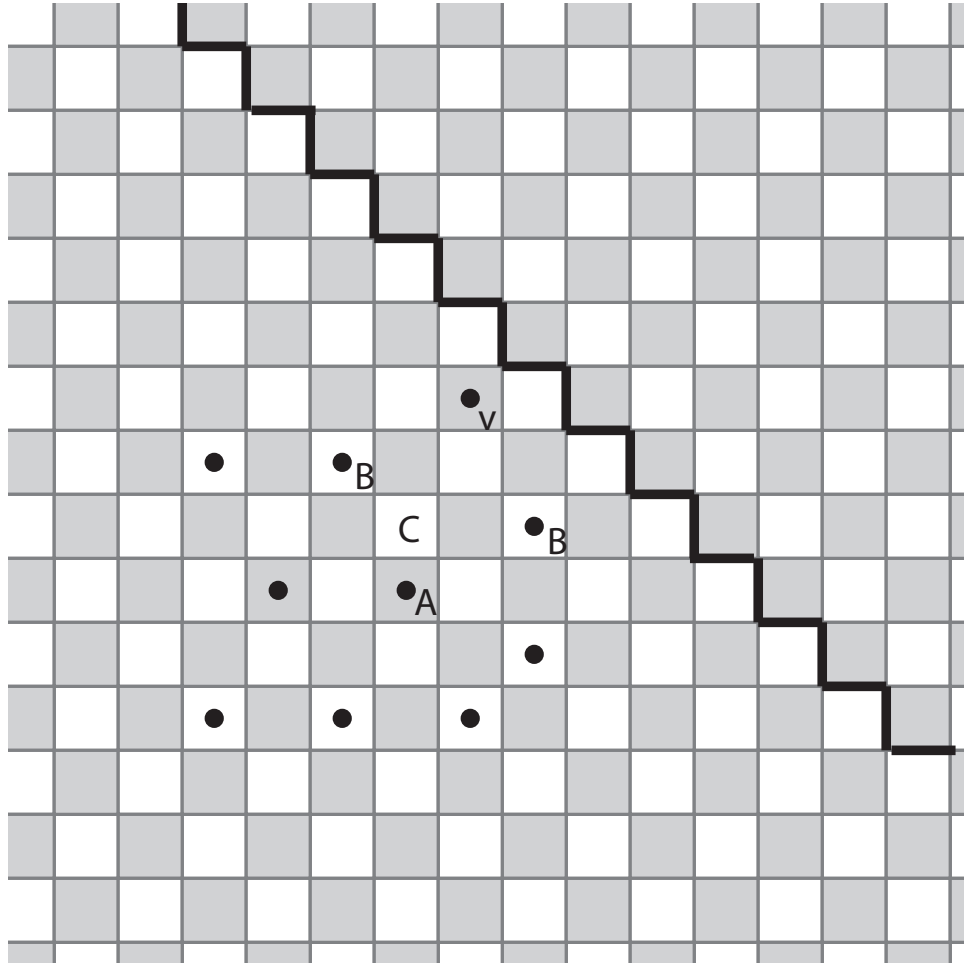


Figure 5: Starting the path \mathbf{P}_1 when v lies near a diagonal edge.

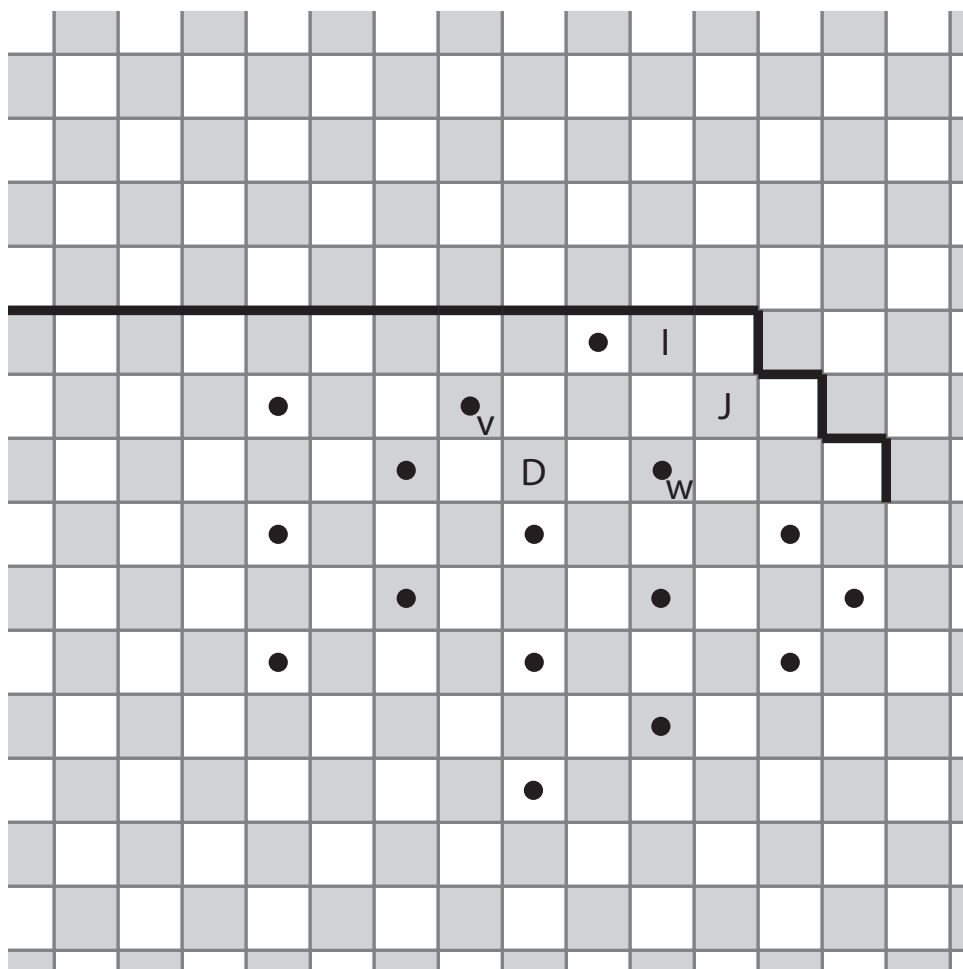


Figure 6: Case 2.

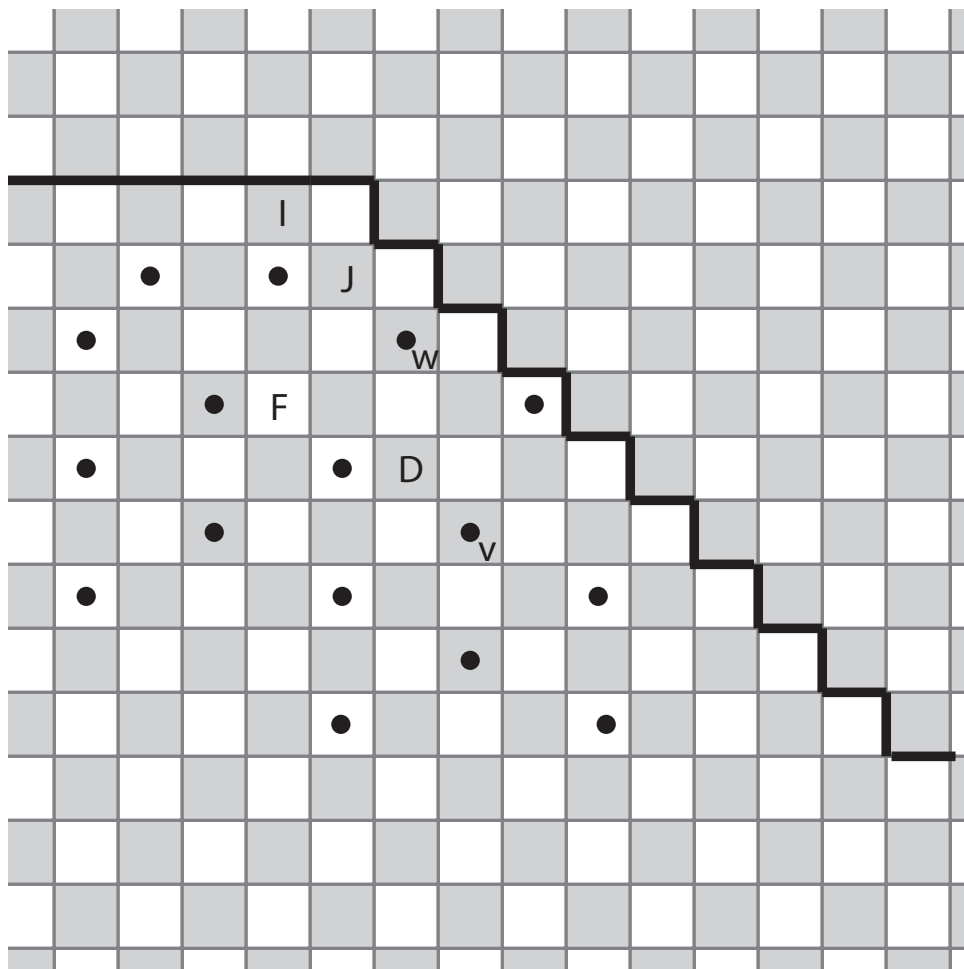


Figure 7: Case 3.

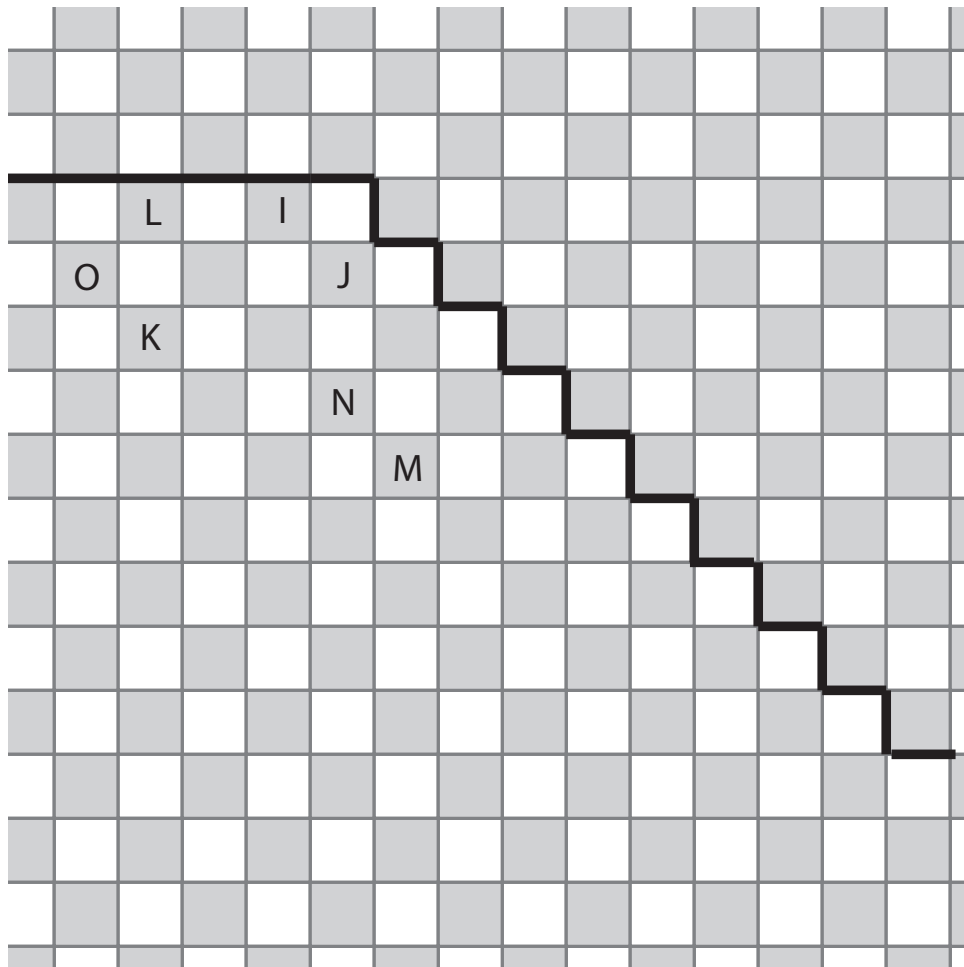


Figure 8: Identifying locations near a corner.

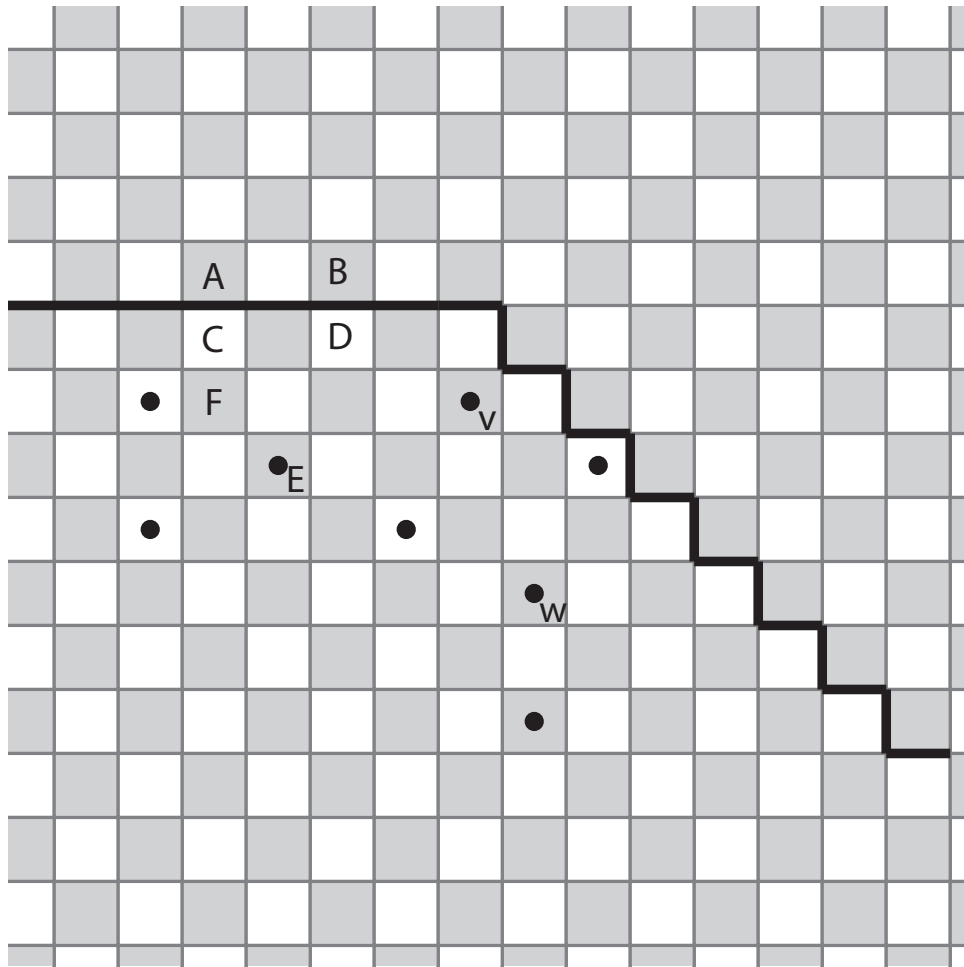


Figure 9: Case 4 (a) (i).



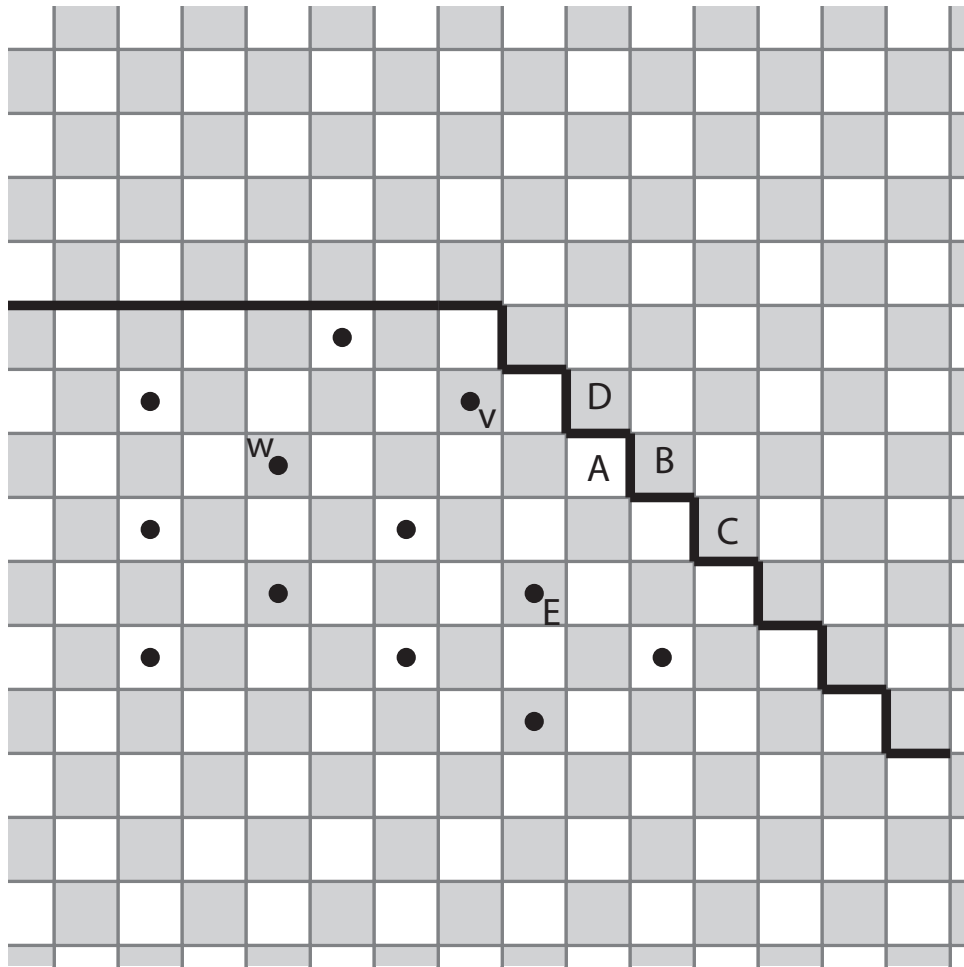


Figure 11: Case 4 (a) (iii).

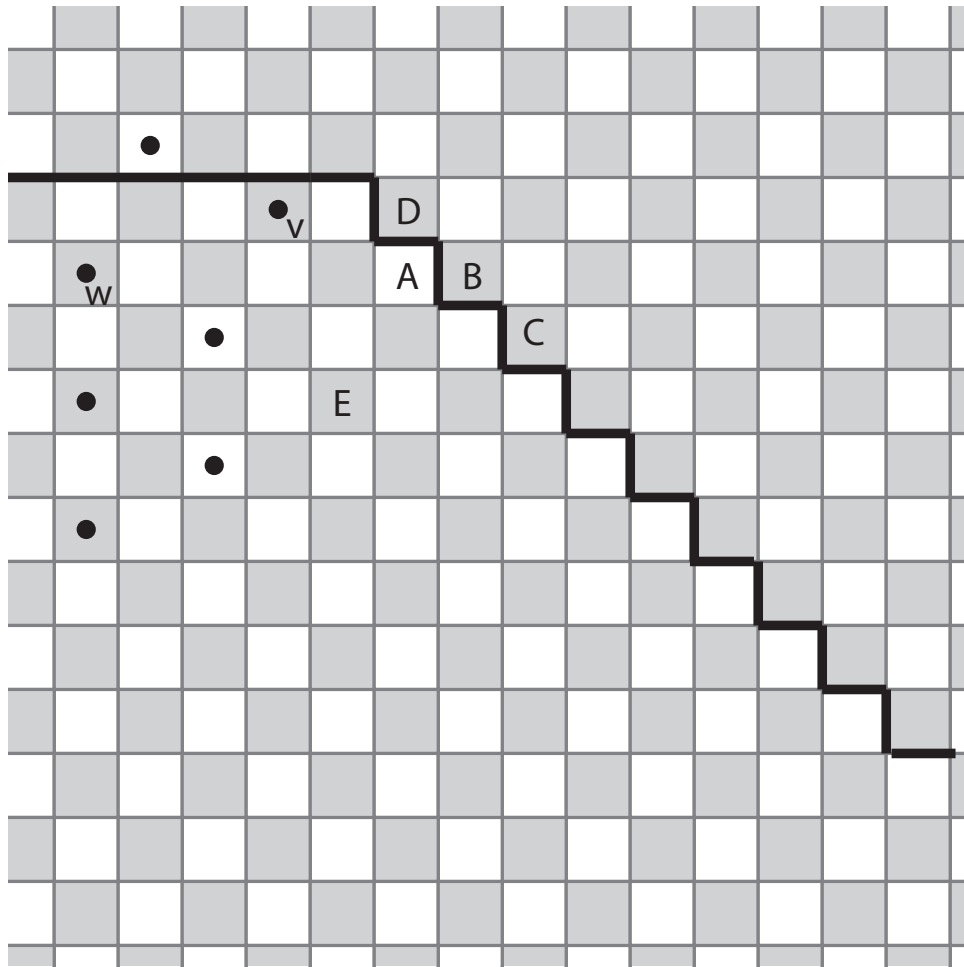


Figure 12: Case 4 (b) (i).

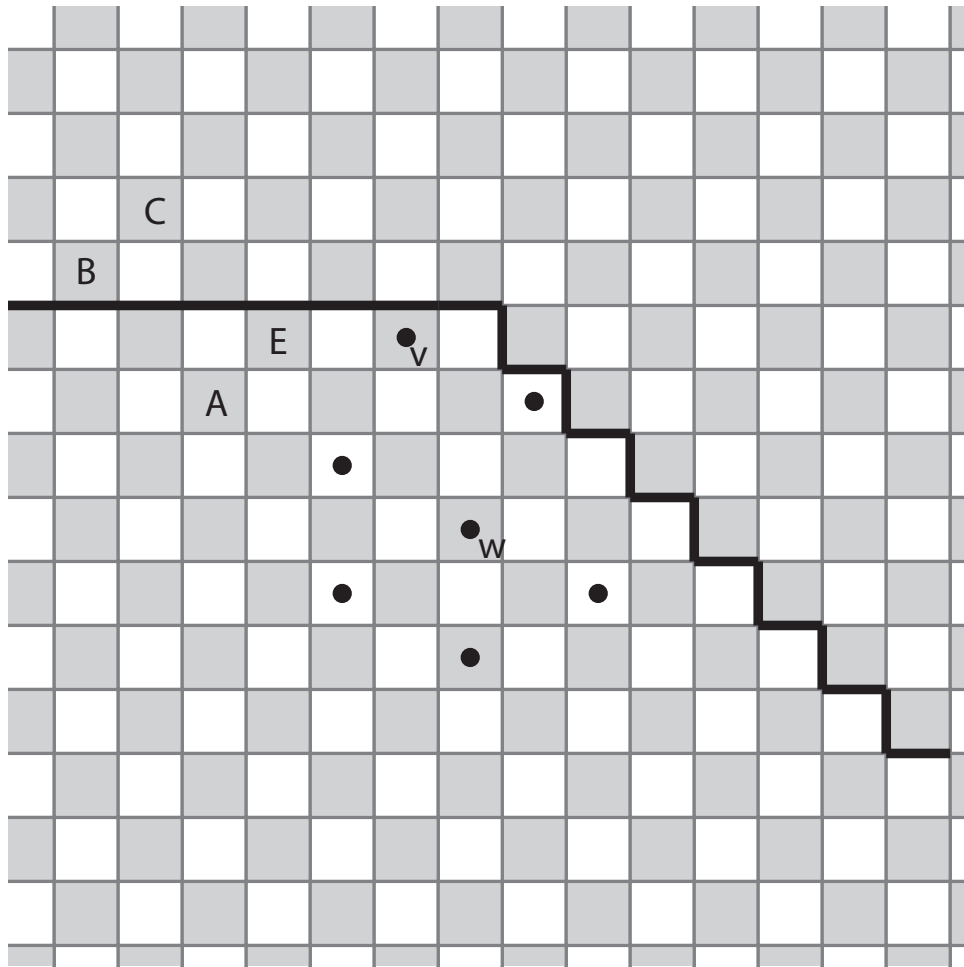


Figure 13: Case 4 (b) (ii).

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